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CAVITATING FLOWS THROUGH A CASCADE OF FLAT PLATES

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SUMMARY

The purpose of this research is to consider the flow through a cascade of bluff bodies, behind which there exist cavities, by using the free streamline theory. When the wake extends to infinity, both the free surface and the velocity on the free surface are unknown and the cavitation number cannot be specified arbitrarily. Given the geometry of the cascade, a numerical method is described in which we obtain the shape of the free surface and the cavitation number. We obtain the relationship between the contraction coefficient, cavitation number and drag coefficient.

KEY WORDS Cascade Cavitating flows

1. INTRODUCTION

The study of cavitation flows is an old subject. The first major step concerning the resistance of an object to a fluid flow was made over a century ago when Kirchhoff[1] introduced an idealized inviscid flow model with free streamlines and employed the conformal mapping technique that had been developed by Helmholtz[2] for treating two-dimensional jets formed by free streamlines. The following decades saw numerous extensions of the complex analysis method, e.g. Levi-Civita,[3] Greenhill, etc. Compendia of this material are those of Birkhoff and Zarantonello,[4] Gilbarg,[5] Woods,[6] Gurevich[7] and Milne-Thomson.[8] An extensive review of the literature may also be found in the exposition by Wu.[9]

The solutions of cavitating flow by employing complex analysis may be divided into two categories. One is the direct problem for the calculation of the cavitating flow passing an obstacle having a solid wall of a prescribed shape. The other method is the inverse problem that is described in an auxiliary plane. In an auxiliary plane the inverse problem is easier to solve than the direct problem because the inverse problem is completely determined. As we know, although the inverse problem is easier to solve, the shape of the obstacle cannot be specified in advance. The reason is that there exists a fundamental difficulty in that for a simple problem the analytic conformal mapping from the physical plane onto an auxiliary plane is unknown. We can specify a function in an auxiliary plane but the shape in the physical plane is unknown until the problem has been completely solved. It is just this feature that causes the difficulty for all except a few very simple cases. In the past the direct problem has been solved through the method for the inverse problem. When we consider the problem in an auxiliary plane in which a parametric variable is taken as an independent variable, the direct problem in the physical plane becomes indirect mathematically and the direct problem in mathematics become indirect in physics. For a direct

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problem the use of an inverse method based on complex analysis is not as convenient as the finite element and finite boundary element methods.

Despite this difficulty, many useful problems have been solved by the inverse method in which the angle between the x-axis and the tangent to the body surface is represented by a power series in the auxiliary variable. For example, Brodetsky[10] treated the problem of the symmetric cavitating flow about circular and elliptic cylinders. Larock and Street[11] expressed the angle of the obstacle as a piecewise linear parametric and also as a polynomial function. The final body profile found as the result of such a calculation cannot be determined in advance unless a very large number of terms in the series is used.

Wu and Wang[12] obtained a formal solution in terms of two non-linear integral equations and two unknown parameters entering the problem, the connection between the physical and potential planes being completely implicit, through the kernels of the integral equations. These equations immediately provide the exact solution of a wide class of inverse problems, while the general direct problems are still difficult to solve exactly. The constants and the implicit relation were determined by a substitution method in a functional iteration. The convergence of the iteration is very sensitive to and dependent upon the choice of a basic flow or starting solution. Furuya[13] used a double iteration loop coupled with Newton's method to improve the convergence. More recent contributions to the problem are by Vanden-Broeck,[14] who solved the cavitating flow problem numerically by series truncation, and by Dias *et al.*[15] and Elcrat and Trefethen,[16] whose computations are based upon numerical techniques for a modified Schwartz–Christoffel mapping.

The linearized free streamline theory of Tulin,[17] for example, is a simple, direct but approximate method for predicting forces on thin bodies. Unfortunately, this theory fails to calculate the pressure distribution correctly.

The main objective of this paper is to exhibit how to directly solve the direct problems of cavitating flows passing a cascade by employing complex analysis. On the basis of the free streamline theory the problem is turned into a mixed boundary value problem and a Hilbert solution is obtained in the auxiliary plane. When we consider the problem in the physical plane, the inverse problem is transformed to a direct problem. When the wake extends to infinity and the condition far upstream and the geometry of the cascade are specified, the cavitation number is unknown. A numerical scheme is presented to obtain the cavitation number, the profile of the free surface, the contraction coefficient and the drag coefficient. The method can be applied to cavitating flows past arbitrarily curved obstacles and to free surface flows with surface tension.

2. FORMULATION

Consider the steady two-dimensional motion of a fluid flowing through a cascade with a grid of normal plates as shown in Figure 1. The fluid is assumed to be inviscid and of constant density ρ, the effects of gravitation and surface tension are neglected and potential flow solutions will be sought. Because of the symmetry of the flow field we need only consider the flow region which is contained between the axis of symmetry and ABCD (Figure 2)

We introduce a co-ordinate system $z=x+iy$ with origin at the separation point C. The free streamline CD is bounded on one side by a region of constant pressure $p=p_0$. The speed is u and the angle that the velocity vector forms with the positive x-axis will be defined as θ. Along the free surface CD we apply Bernoulli's equation in the form

$$U^2/2+p_\infty/\rho = u_0^2/2+p_0/\rho \tag{1}$$

or

$$u_0/U = \sqrt{(\sigma+1)}, \tag{2}$$

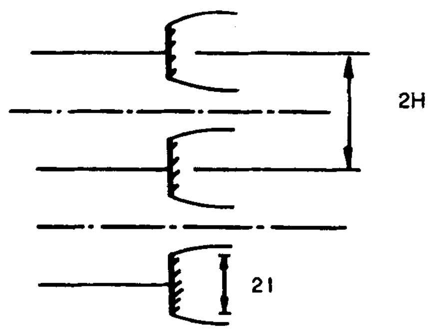

Figure 1. The geometry

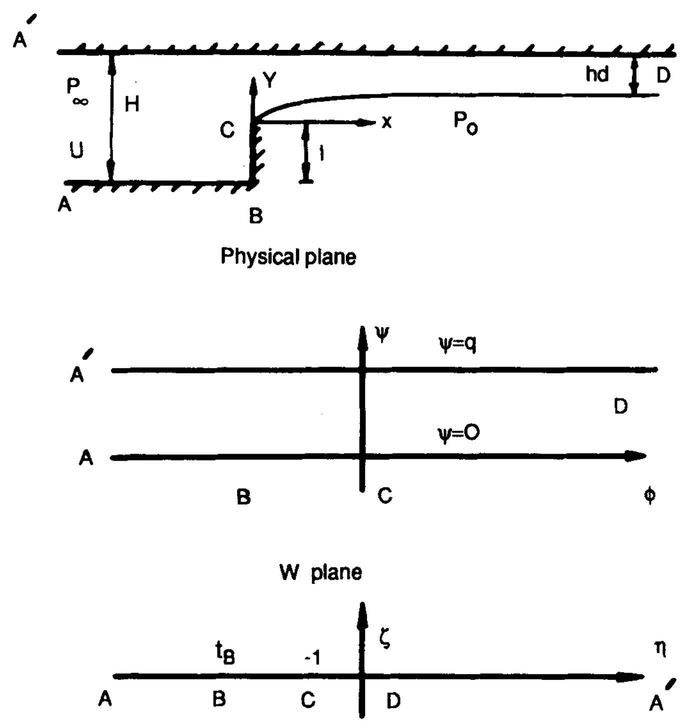

Figure 2. The transformation

in which $\sigma = 2(p_\infty - p_0)/\rho U^2$ is the cavitation number, U is the fluid speed upstream and q is the volumetric flux per unit breadth passing between the streamline ABCD and the axis of symmetry.

The conservation of mass requires that $q = HU = h_d u_0$. Putting $u_0/U = H/h_d$, expression (2) yields

$$\sigma = (H/h_d)^2 - 1. \tag{3}$$

As we can see, when the condition upstream and the geometry of the plate are specified, the cavitation number and the velocity on the free surface are unknown because the profile of the free surface is unknown. Defining the complex velocity potential $W=\phi+i\psi$, where ϕ is the velocity potential and ψ is the streamfunction, the complex velocity is

$$\mathrm{d}W/\mathrm{d}z = u\mathrm{e}^{-i\theta} \tag{4}$$

and the logarithm of the complex velocity is

$$\Omega = \ln\left(\frac{1}{U}\frac{\mathrm{d}W}{\mathrm{d}z}\right) = \tau - \mathrm{i}\theta, \tag{5}$$

where $\tau = \ln(u/U)$ and τ, $\mathrm{d}W/\mathrm{d}z$ and Ω will be analytic functions of W. Without loss of generality we may define the streamfunction $\psi=0$ to be the streamline ABCD and choose $\phi=0$ to pass through the separation point C. The strip in the W-plane is mapped onto the upper half-plane of the auxiliary plane t by

$$t = -\mathrm{e}^{-\pi W/q} \quad \text{or} \quad W = f(t) = -(q/\pi)\ln(-t). \tag{6}$$

W, $\mathrm{d}W/\mathrm{d}z$ and Ω will be analytical functions of t, and the boundary conditions on the real η-axis of the t-plane are

$$\begin{aligned}
&\operatorname{Im}\Omega(\eta) = 0, && \eta < t_{\mathrm{B}},\\
&\operatorname{Im}\Omega(\eta) = -\pi/2, && t_{\mathrm{B}} < \eta < -1,\\
&\operatorname{Re}\Omega(\eta) = \tau(\eta) = \ln[\sqrt{(\sigma+1)}], && -1 < \eta < 0,\\
&\operatorname{Im}\Omega(\eta) = 0, && 0 < \eta.
\end{aligned} \tag{7}$$

The above is a mixed boundary value problem in the upper half-plane. By referring to the general solution of the Riemann–Hilbert problem[18] we obtain the solution of Ω in the form

$$\Omega(t) = \frac{\sqrt{[t(t+1)]}}{\pi}\left(\int_{t_{\mathrm{B}}}^{-1}\frac{-\pi/2}{\sqrt{[\eta(\eta+1)]}(\eta-t)}\,\mathrm{d}\eta + \int_{-1}^{0}\frac{\tau}{\sqrt{[-\eta(\eta+1)]}(\eta-t)}\,\mathrm{d}\eta\right), \tag{8}$$

where $X(t)=\sqrt{[t(t+1)]}$ is a homogeneous solution and we choose a branch cut for $X(t)$ such that

$$\begin{aligned}
&X^{+}(\eta) = \sqrt{[\eta(\eta+1)]}, && \eta < -1,\\
&X^{+}(\eta) = -\mathrm{i}\sqrt{[-\eta(\eta+1)]}, && -1 < \eta < 0,\\
&X^{+}(\eta) = -\sqrt{[\eta(\eta+1)]}, && 0 < \eta.
\end{aligned} \tag{9}$$

By taking the boundary value on the real axis η and separating the real and imaginary parts of Ω, one gets the velocity on the solid wall,

$$\tau(t) = \ln\left(\frac{u(t)}{U}\right) = \ln\left(\frac{\sqrt{|t-t_{\mathrm{B}}|}}{\sqrt{[(t_{\mathrm{B}}+1)t]}+\sqrt{[t_{\mathrm{B}}(t+1)]}}\right) + \ln(\sigma+1) \tag{10}$$

or

$$u(t) = U\left(\frac{\sqrt{|t-t_{\mathrm{B}}|}}{\sqrt{[(t_{\mathrm{B}}+1)t]}+\sqrt{[t_{\mathrm{B}}(t+1)]}}\right)\sqrt{(\sigma+1)}, \tag{11}$$

and the direction on the free surface,

$$\theta(t) = \frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\frac{(t_{\mathrm{B}}+1)t+(t+1)t_{\mathrm{B}}}{t_{\mathrm{B}}-t}\right). \tag{12}$$

The boundary condition at infinity is

$$\lim_{t \to \infty} u(t) = U. \tag{13}$$

Applying (11) to (13) we obtain

$$\sqrt{(-t_B - 1)} + \sqrt{(-t_B)} = \sqrt{(\sigma + 1)}. \tag{14}$$

2.1. *Inverse problem*

For an inverse problem the solution is completely determined by the parameter t_B or by the cavitation number. For example, when t_B is specified, from (14) we get the cavitation number and from (11) we obtain the velocity on the free surface. The physical plane including the profiles of the solid wall and free surface is given by

$$z(t) = \int \frac{e^{i\theta}}{u(\eta)} \frac{dW}{d\eta} d\eta \tag{15}$$

and the length l of plate can be obtained from

$$l = \frac{H}{\pi} \int_{t_B}^{-1} e^{-i\tau(\eta)} \frac{d\eta}{\eta}. \tag{16}$$

Although the inverse problem is completely determined by the parameter t_B or by σ, we do not know the length of the plate until the problem is completely solved.

2.2. *Direct problem*

We now turn to the direct problem in which H, l, U and p_∞ are specified in advance. We will seek the profile of the free surface and the velocity on the plate for the calculation of the drag coefficient. From (3) we know that the cavitation number will be unknown owing to h_d being unknown and the velocity on the free surface is also unknown from (2).

Now let us consider the problem in the physical plane. In fact, for a direct problem the unknown functions are the velocity and potential function on the plate and the profile of the free surface and the velocity u_0 on the free surface. But all these functions can be taken as functions of the arc length s of the streamline, namely $u(s)$, $\phi(s)$ on the wall, $x(s)$, $y(s)$, $\theta(s)$ on the free surface, and these functions are related by

$$d\phi(s)/ds = u(s) \quad \text{on the wall and free surface,} \tag{17}$$

$$\left.\begin{aligned} dx(s)/ds &= \cos\theta(s) \\ dy(s)/ds &= \sin\theta(s) \end{aligned}\right\} \quad \text{on the free surface.} \tag{18}$$

In the physical plane we can rewrite (11) and (12) as

$$u(s) = U\left(\frac{\sqrt{|t(s) - t(B)|}}{\sqrt{[(t(B)+1)t(s)]} + \sqrt{[t(B)(t(s)+1)]}}\right)\sqrt{(\sigma+1)}, \tag{19}$$

$$\theta(s) = \frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\frac{(t(B)+1)t(s) + (t(s)+1)t(B)}{t(B) - t(s)}\right), \tag{20}$$

where $t(s) = -e^{-\pi\phi(s)/q}$, $t(B) = -e^{-\pi\phi(B)/q}$, $\phi(B)$ is the potential function at the point B and from (17) and (18) we obtain the potential function

$$\phi(l) = \int_0^l u(s)\mathrm{d}s \quad \text{on the free surface and the plate,} \tag{21}$$

and the profile of the free surface is given by

$$x(l) = \int_0^l \cos\theta(s)\,\mathrm{d}s, \qquad y(l) = \int_0^l \sin\theta(s)\,\mathrm{d}s. \tag{22}$$

The drag coefficient on the plate, normalized by the plate length, can now be defined by

$$C_\mathrm{D} = 2P/\rho U^2 l, \tag{23}$$

where $P = \int (p - p_0)\,\mathrm{d}l$ is the force on the plate. Applying the Bernoulli equation on the free surface and the plate surface we can write (23) as

$$C_\mathrm{D} = \sigma + 1 - \frac{1}{l}\int \frac{u^2(s)}{U^2}\,\mathrm{d}s, \tag{24}$$

where $u(s)$ is the velocity on the plate. The contraction coefficient is defined by

$$C_\mathrm{C} = h_\mathrm{d}/(H - l). \tag{25}$$

3. NUMERICAL METHOD FOR THE DIRECT PROBLEM

For a specified length of plate the nodes are distributed on the plate and free surface in a similar manner to that used in the finite element and boundary element methods. We briefly outline the procedure of the numerical method.

(a) We assume a cavitation number σ^0 and obtain the velocity u_0^0 and the potential function $\phi^0(s)$ on the free surface from (2) and (21) respectively. Assume the velocity $u^0(s)$ on the plate and obtain the potential function $\phi^0(s)$ from (21).
(b) Putting $\phi^0(s)$ into the right-hand sides of equations (19) and (20) we obtain the velocity $u^1(s)$ on the plate and the direction $\theta^1(s)$ on the free surface.
(c) From (22) an approximate profile of the free surface is determined in addition to obtaining h_d^1.
(d) Putting h_d^1 into (3) yields a new cavitation number.

These values are used as new approximations and the iterative procedure from step (b) to step (d) is repeated until

$$\left|\frac{\sigma^n - \sigma^{n-1}}{\sigma^n}\right| < \varepsilon_1, \qquad \left|\frac{\phi^n(\mathrm{B}) - \phi^{n-1}(\mathrm{B})}{\phi^n(\mathrm{B})}\right| < \varepsilon_2, \tag{26}$$

where ε_1 and ε_2 are two preassigned small quantities.

4. NUMERICAL RESULTS

For a given geometry of the cascade, i.e. H/l, the mesh nodes are distributed according to the change of the velocity on the plate and the direction on the free surface. In the neighbourhood of the separation point C the change is rapid and therefore the intervals are smaller than elsewhere. On the plate 41 nodes are used and on the free surface 201 mesh nodes are distributed. The length of the first interval on the plate from C is $\Delta s_1 = 0.001458l$ and Δs_n increases as $\Delta s_n = a\Delta s_{n-1}$ ($a = 1{\cdot}25$) for $n \leqslant 19$ and decreases as $\Delta s_n = \Delta s_{n-1}/a$ for $n \geqslant 20$. The total length calculated for the free surface is $5{\cdot}5(H-l)$, the length of the first interval from C is $\Delta s_1 = 0{\cdot}001\ (H-l)$ and Δs_n increases as $\Delta s_n = b\Delta s_{n-1}$ ($b = 1{\cdot}02495$). We found that the initial approximation to the cavitation number σ^0 has little influence on the convergence; in the calculations presented here we set $\sigma^0 = 15$. All the numerical results presented in this paper were obtained using double-precision arithmetic on the Amdahl 5860 computer at Leeds University and convergent results were always obtained using fewer than seven iterations and less than 10 s of CPU time. The results for the cavitation number, contraction coefficient, drag coefficient and u_0/U are given in Table I and the computed profiles of the free surfaces are sketched in Figure 3. In all the calculations presented here we have found $\varepsilon_1 = \varepsilon_2 = 10^{-6}$ to be sufficiently small so that all the results presented are accurate to the number of decimals quoted.

The results show that as the ratio H/l decreases, the cavitation number σ, relative velocity u_0/U and drag coefficient increase but the contraction coefficient decreases. As $H/l \to 1$, i.e.

Table I. Computed results

H/l	σ	C_C	u_0/U	C_D
1·0500	1178·5	0·61145	34·345	1167·6
1·1000	322·01	0·61197	17·975	316·87
1·5000	22·057	0·62477	4·8018	21·665
2·0000	8·6368	0·64427	3·1043	8·8478
3·0000	3·9031	0·67742	2·2149	4·4182
4·0000	2·6020	0·70254	1.8979	3·2205
5·0000	1·9962	0.72215	1·7310	2·6676
6·0000	1·6439	0·73800	1·6260	2·3478
7·0000	1·4122	0·75118	1·5531	2·1383
8·0000	1·2473	0·76237	1·4991	1·9896
9·0000	1·1234	0·77204	1·4572	1·8781
10·000	1·0266	0·78051	1·4236	1·7911

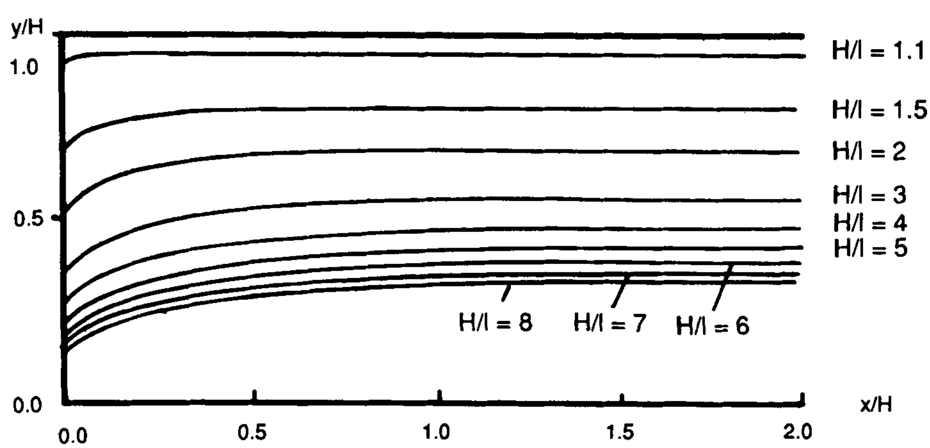

Figure 3. Profiles of free surfaces

$l/(H-l) \to \infty$, the contraction coefficient limits the results of flow from an infinite reservoir:

$$C_C = \frac{\pi}{\pi+2} = 0{\cdot}61104. \tag{27}$$

5. CONCLUSIONS

In this paper we have described in detail the numerical method for the direct problem of the flow through a cascade of bluff bodies behind which there exist cavities whose wake is of infinite extent. For a direct problem the cavitation number, the velocity on the free surface and the profile of the free surface must be solved. Because we consider the problem in the physical plane, the inverse problem is transformed into a direct problem on which the shape of the obstacle may be specified in advance, so we avoid the difficulty that the physical plane is not explicitly provided until the whole problem is solved. As a result, the nodes may be distributed in a similar manner to that in the finite element and boundary element methods. The method used in this paper can be extended to calculate cavitating flows passing through arbitrarily shaped obstacles and, with a certain amount of modification, the method may be adapted to include the effects of surface tension. The calculation of cavitating flows passing arbitrarily curved obstacles under the action of surface tension will be reported in a later paper. In this paper we have used a boundary integral equation method. Although no comparative results exist for using finite difference and finite element methods, we would expect the present method to be superior to these methods since only the boundary of the solution domain requires discretization and convergent results were always obtained using less than seven iterations.

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REFERENCES

1. G. Kirchhoff, 'Zur theorie freier flüssigkeitsstrahlen', *J. Reine Angew. Math.*, **70**, 289–298 (1869).
2. H. Helmholtz, 'On discontinuous movements of fluids', *Phil. Mag.*, **36**, 337–346 (1868).
3. T. Levi-Civita, 'Scie e leggi di resistenzia', *R. C. Cir. Mat. Palermo*, **18**, 1–13 (1907).
4. G. Birkhoff and E. H. Zarantonello, *Jet, Wakes and Cavities*, Academic Press, New York, 1957.
5. D. Gilbarg, 'Jets and cavities', *Handb. Phys.*, **9**, 311–445 (1960).
6. L. C. Woods, *The Theory of Subsonic Plane Flow*, Cambridge University Press, Cambridge, 1961.
7. M. I. Gurevich, *The Theory of Jets in Ideal Fluid*, Academic Press, New York, 1965.
8. L. Milne-Thomson, *Theoretical Hydrodynamics*, Macmillan, London, 1971.
9. T. Y. Wu, 'Cavity and wake flows', *Ann. Rev. Fluid Mech.*, **4**, 243–284 (1972).
10. S. Brodetsky, 'Discontinuous fluid motion past circular and elliptic cylinders', *Proc. R. Soc. A*, **102**, 542–553 (1922).
11. B. Larock and R. Street, 'Cambered bodies in cavitating flow—a nonlinear analysis and design procedure', *J. Ship Res.*, **12**, 1–13 (1968).
12. T. Y. Wu and D. P. Wang, 'A wake model for free-streamline flow theory, Part 2, Cavity flows past obstacles of arbitrary profile', *J. Fluid Mech.*, **18**, 65–93 (1964).
13. O. Furuya, 'Nonlinear calculation of arbitrarily shaped supercavitating hydrofoils near a free surface', *J. Fluid Mech.*, **68**, pt. 1, 21–40 (1975).
14. J.-M. Vanden-Broeck, 'The influence of surface tension on cavitating flow past a curved obstacle', *J. Fluid Mech.*, **133**, 255–264 (1983).
15. F. Dias, A. R. Elcrat and L. N. Trefethen, 'Ideal jet flow in two dimensions', *J. Fluid Mech.*, **185**, 275–288 (1987).
16. A. R. Elcrat and L. N. Trefethen, 'Free-streamline flow over a polygonal obstacle', *J. Comput. Appl. Math.*, **14**, 251–265 (1986).
17. M. Tulin, 'Supercavitating flows—small perturbation theory', *J. Ship Res.*, **7**, 16–37 (1964).
18. N. I. Muskhelishivili, *Singular Integral Equations*, P. Noordhoff N. V., Groningen, Holland, 1953.